1 Local Stabilization with Linear State Feedback

1. Plant Model

$$\dot{x} = f(x,u) \qquad \begin{cases} u \in R^m \\ x \in R^n \end{cases}$$

2. Equilibrium Conditions

$$0 = f(\bar{x},\bar{u})$$

3. Linear Approximation

$$A = \left.\frac{\partial f}{\partial x}\right|_{(\bar{x},\bar{u})} \qquad B = \left.\frac{\partial f}{\partial u}\right|_{(\bar{x},\bar{u})}$$

4. Design Model

$$\dot{\tilde{x}} \approx A\tilde{x} + B\tilde{u} \qquad \begin{cases} \tilde{u} = u - \bar{u} \\ \tilde{x} = x - \bar{x} \end{cases}$$

5. Design Calculations
   (a) Assume (A,B) is stabilizable.
   (b) Compute K such that $A - BK$ is Hurwitz.
   (c) Employ the following controller:

$$\tilde{u} = -K\tilde{x}$$

$$\Updownarrow$$

$$u = \bar{u} - K(x - \bar{x})$$

   (d) Note that $(\bar{u},\bar{x})$ must be known for implementation.
   (e) Note that x must be measured for implementation.

6. Closed-Loop Analysis
   (a) Translation of Coordinates

$$\tilde{x} = x - \bar{x}$$

   (b) Nonlinear Dynamics

$$\dot{\tilde{x}} = F(\tilde{x}) = f(\bar{x} + \tilde{x}, \bar{u} - K\tilde{x})$$

   (c) Equilibrium at Origin

$$\tilde{x} = 0 \quad \Rightarrow \quad \dot{\tilde{x}} = 0$$

   (d) Jacobian Matrix Evaluation

$$\left.\frac{\partial F}{\partial \tilde{x}}\right|_0 = \left.\frac{\partial f}{\partial x}\right|_{(\bar{x},\bar{u})}(I) + \left.\frac{\partial f}{\partial u}\right|_{(\bar{x},\bar{u})}(-K) = A - BK$$

(e) The Jacobian matrix

$$A_{\text{cl}} = A - BK$$

is Hurwitz by design (see step 5b). According to Lyapunov's Indirect Method, $\tilde{x} = 0$ is an exponentially stable equilibrium point of the closed-loop system.

(f) Domain of Attraction Estimate

i. Solve the Lyapunov equation

$$A_{\text{cl}}^T P + P A_{\text{cl}} = -Q$$

for some $Q = Q^T > 0$ to obtain $P = P^T > 0$.

ii. Define

$$V(\tilde{x}) = \tilde{x}^T P \tilde{x}$$

and

$$\Omega = \{\tilde{x} \in R^n : V(\tilde{x}) \leq c\}$$

where $c > 0$ is chosen small enough to guarantee that $\dot{V}(\tilde{x}) < 0$ in $\Omega \setminus \{0\}$.

iii. The domain of attraction contains Ω, i.e. $\tilde{x}(0) \in \Omega$ implies that $\tilde{x}(t) \to 0$ as $t \to \infty$.

2 Local Stabilization with Linear Output Feedback

1. Plant Model

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x) \end{aligned} \qquad \begin{cases} u \in R^m \\ x \in R^n \\ y \in R^p \end{cases}$$

2. Equilibrium Conditions

$$\begin{aligned} 0 &= f(\bar{x},\bar{u}) \\ \bar{y} &= h(\bar{x}) \end{aligned}$$

3. Linear Approximation

$$A = \left.\frac{\partial f}{\partial x}\right|_{(\bar{x},\bar{u})} \qquad B = \left.\frac{\partial f}{\partial u}\right|_{(\bar{x},\bar{u})} \qquad C = \left.\frac{\partial h}{\partial x}\right|_{\bar{x}}$$

4. Design Model

$$\begin{aligned} \dot{\tilde{x}} &\approx A\tilde{x} + B\tilde{u} \\ \tilde{y} &\approx C\tilde{x} \end{aligned} \qquad \begin{cases} \tilde{u} = u - \bar{u} \\ \tilde{x} = x - \bar{x} \\ \tilde{y} = y - \bar{y} \end{cases}$$

5. Design Calculations

   (a) Assume (A,B) is stabilizable and (A,C) is detectable.

   (b) Compute K such that $A - BK$ is Hurwitz and H such that $A - HC$ is Hurwitz.

   (c) Employ the following controller:

$$\begin{aligned} \tilde{u} &= -K\hat{\tilde{x}} \\ \dot{\hat{\tilde{x}}} &= A\hat{\tilde{x}} + B\tilde{u} - H(C\hat{\tilde{x}} - \tilde{y}) \end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned} u &= \bar{u} - K\hat{\tilde{x}} \\ \dot{\hat{\tilde{x}}} &= A\hat{\tilde{x}} + B(u-\bar{u}) - H(C\hat{\tilde{x}} - (y-\bar{y})) \end{aligned}$$

   (d) Note that $(\bar{u},\bar{y})$ must be known for implementation.

   (e) Note that y must be measured for implementation.

6. Closed-Loop Analysis

   (a) Translation of Coordinates

$$\begin{aligned} \tilde{x} &= x - \bar{x} \\ \tilde{e} &= \hat{\tilde{x}} - \tilde{x} \end{aligned}$$

   (b) Nonlinear Dynamics

$$\begin{aligned} \dot{\tilde{x}} &= F_1(\tilde{x},\tilde{e}) = f(\bar{x}+\tilde{x}, \bar{u} - K(\tilde{x}+\tilde{e})) \\ \dot{\tilde{e}} &= F_2(\tilde{x},\tilde{e}) = (A - BK - HC)(\tilde{x}+\tilde{e}) + H(h(\bar{x}+\tilde{x}) - \bar{y}) - f(\bar{x}+\tilde{x}, \bar{u} - K(\tilde{x}+\tilde{e})) \end{aligned}$$

(c) Equilibrium at Origin

$$\begin{bmatrix} \tilde{x} \\ \tilde{e} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{e}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(d) Jacobian Matrix Evaluation

$$\begin{aligned}
\left.\frac{\partial F_1}{\partial \tilde{x}}\right|_{(0,0)} &= \left.\frac{\partial f}{\partial x}\right|_{(\bar{x},\bar{u})} (I) + \left.\frac{\partial f}{\partial u}\right|_{(\bar{x},\bar{u})} (-K) = A - BK \\
\left.\frac{\partial F_1}{\partial \tilde{e}}\right|_{(0,0)} &= \left.\frac{\partial f}{\partial u}\right|_{(\bar{x},\bar{u})} (-K) = -BK \\
\left.\frac{\partial F_2}{\partial \tilde{x}}\right|_{(0,0)} &= A - BK - HC + H \left.\frac{\partial h}{\partial x}\right|_{\bar{x}} (I) - \left.\frac{\partial f}{\partial x}\right|_{(\bar{x},\bar{u})} (I) - \left.\frac{\partial f}{\partial u}\right|_{(\bar{x},\bar{u})} (-K) = 0 \\
\left.\frac{\partial F_2}{\partial \tilde{e}}\right|_{(0,0)} &= A - BK - HC - \left.\frac{\partial f}{\partial u}\right|_{(\bar{x},\bar{u})} (-K) = A - HC
\end{aligned}$$

(e) The Jacobian matrix

$$A_{\text{cl}} = \begin{bmatrix} A - BK & -BK \\ 0 & A - HC \end{bmatrix}$$

is block diagonal, so its eigenvalues are those of $A - BK$ and $A - HC$ combined. Thus, A_{cl} is Hurwitz by design (see step 5b). According to Lyapunov's Indirect Method, $(\tilde{x}, \tilde{e}) = (0, 0)$ is an exponentially stable equilibrium point of the closed-loop system.

(f) Domain of Attraction Estimate

i. Solve the Lyapunov equation

$$A_{\text{cl}}^T P + P A_{\text{cl}} = -Q$$

for some $Q = Q^T > 0$ to obtain $P = P^T > 0$.

ii. Define

$$V(\tilde{z}) = \tilde{z}^T P \tilde{z}, \ \tilde{z} = \begin{bmatrix} \tilde{x} \\ \tilde{e} \end{bmatrix}$$

and

$$\Omega = \{\tilde{z} \in R^{2n} : V(\tilde{z}) \leq c\}$$

where $c > 0$ is chosen small enough to guarantee that $\dot{V}(\tilde{z}) < 0$ in $\Omega \setminus \{0\}$.

iii. The domain of attraction contains Ω, i.e. $\tilde{z}(0) \in \Omega$ implies that $\tilde{z}(t) \to 0$ as $t \to \infty$.

3 Local Output Regulation with Linear State Feedback via Integral Control

1. Plant Model

$$\begin{aligned}\dot{x} &= f(x,u)\\ y &= h(x)\end{aligned} \qquad \begin{cases} u \in R^m \\ x \in R^n \\ y \in R^m \end{cases} \qquad \begin{array}{c}\text{Design Goal:}\\ \lim_{t\to\infty} y = r\end{array}$$

2. Equilibrium Conditions

$$\begin{aligned}0 &= f(\bar{x},\bar{u})\\ r &= h(\bar{x})\end{aligned}$$

3. Linear Approximation

$$A = \left.\frac{\partial f}{\partial x}\right|_{(\bar{x},\bar{u})} \qquad B = \left.\frac{\partial f}{\partial u}\right|_{(\bar{x},\bar{u})} \qquad C = \left.\frac{\partial h}{\partial x}\right|_{(\bar{x})}$$

4. Design Calculations

   (a) Assume that (A,B) is stabilizable, and that the square matrix

   $$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$$

   is nonsingular.

   (b) Compute $\mathcal{K}$ such that $\mathcal{A}-\mathcal{B}\mathcal{K}$ is Hurwitz, where

   $$\mathcal{A} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \quad \mathcal{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} \quad \mathcal{K} = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

   (c) Employ the following controller:

   $$\begin{aligned}u &= -K_1 x - K_2\sigma\\ \dot{\sigma} &= y - r\end{aligned}$$

   (d) Note that nominal values of (f,h) parameters may be used for design calculations.

   (e) Note that equilibrium data are not required for implementation.

   (f) Note that x and y must be measured for implementation.

5. Closed-Loop Analysis

   (a) Nonlinear Dynamics

   $$\begin{aligned}\dot{x} &= F_1(x,\sigma) = f(x,-K_1x-K_2\sigma)\\ \dot{\sigma} &= F_2(x,\sigma) = h(x) - r\end{aligned}$$

   (b) Equilibrium Constraints

   $$\begin{aligned}0 &= f(\bar{x},-K_1\bar{x}-K_2\bar{\sigma})\\ 0 &= h(\bar{x}) - r\end{aligned}$$

(c) For any fixed r of interest, the equilibrium conditions of step 2 are assumed to provide unique corresponding equilibrium values for $\bar{x}$ and $\bar{u}$, via solution of $n+m$ constraints on $n+m$ variables. The design hypotheses of step 4a guarantee that K_2 is a nonsingular matrix. Consequently, for the given r there is a unique equilibrium state $(\bar{x}, \bar{\sigma})$ such that

$$\bar{\sigma} = -K_2^{-1}\left(\bar{u} + K_1\bar{x}\right)$$

At this equilibrium state, the output matches the desired setpoint, i.e. $y = r$. Perturbations of (f, h) parameters affect the equilibrium state but do not destroy the desired property of zero steady-state error at the output.

(d) Jacobian Matrix Evaluation

$$\begin{aligned}
\left.\frac{\partial F_1}{\partial x}\right|_{(\bar{x},\bar{\sigma})} &= \left.\frac{\partial f}{\partial x}\right|_{(\bar{x},\bar{u})}(I) + \left.\frac{\partial f}{\partial u}\right|_{(\bar{x},\bar{u})}(-K_1) = A - BK_1 \\
\left.\frac{\partial F_1}{\partial \sigma}\right|_{(\bar{x},\bar{\sigma})} &= \left.\frac{\partial f}{\partial u}\right|_{(\bar{x},\bar{u})}(-K_2) = -BK_2 \\
\left.\frac{\partial F_2}{\partial x}\right|_{(\bar{x},\bar{\sigma})} &= \left.\frac{\partial h}{\partial x}\right|_{(\bar{x})}(I) = C \\
\left.\frac{\partial F_2}{\partial \sigma}\right|_{(\bar{x},\bar{\sigma})} &= 0
\end{aligned}$$

(e) The Jacobian matrix

$$A_{\text{cl}} = \begin{bmatrix} A - BK_1 & -BK_2 \\ C & 0 \end{bmatrix} = \mathcal{A} - \mathcal{B}\mathcal{K}$$

is Hurwitz by design (see step 4b). According to Lyapunov's Indirect Method, $(x, \sigma) = (\bar{x}, \bar{\sigma})$ is an exponentially stable equilibrium point of the closed-loop system. At this equilibrium point, $y = r$ as desired.

4 Local Output Regulation with Linear Output Feedback via Integral Control

1. Plant Model

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x) \end{aligned} \qquad \begin{cases} u \in R^m \\ x \in R^n \\ y \in R^m \end{cases} \qquad \begin{matrix}\text{Design Goal:} \\ \lim_{t\to\infty} y = r\end{matrix}$$

2. Equilibrium Conditions

$$\begin{aligned} 0 &= f(\bar{x},\bar{u}) \\ r &= h(\bar{x}) \end{aligned}$$

3. Linear Approximation

$$A = \left.\frac{\partial f}{\partial x}\right|_{(\bar{x},\bar{u})} \qquad B = \left.\frac{\partial f}{\partial u}\right|_{(\bar{x},\bar{u})} \qquad C = \left.\frac{\partial h}{\partial x}\right|_{(\bar{x})}$$

4. Design Calculations

   (a) Assume that (A,B) is stabilizable and (A,C) is detectable, and that the square matrix

   $$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$$

   is nonsingular.

   (b) Compute $\mathcal{K}$ such that $\mathcal{A}-\mathcal{B}\mathcal{K}$ is Hurwitz and H such that $A-HC$ is Hurwitz, where

   $$\mathcal{A} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \quad \mathcal{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} \quad \mathcal{K} = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

   (c) Employ the following controller:

   $$\begin{aligned} u &= -K_1\hat{x} - K_2\sigma \\ \dot{\sigma} &= y - r \\ \dot{\hat{x}} &= A\hat{x} + Bu - H(C\hat{x} - y) \end{aligned}$$

   (d) Note that nominal values of (f,h) parameters may be used for design calculations.

   (e) Note that equilibrium data are not required for implementation.

   (f) Note that y must be measured for implementation.

5. Closed-Loop Analysis

   (a) Nonlinear Dynamics

   $$\begin{aligned} \dot{x} &= F_1(x,\sigma,\hat{x}) = f(x, -K_1\hat{x} - K_2\sigma) \\ \dot{\sigma} &= F_2(x,\sigma,\hat{x}) = h(x) - r \\ \dot{\hat{x}} &= F_3(x,\sigma,\hat{x}) = (A - BK_1 - HC)\hat{x} - BK_2\sigma + Hh(x) \end{aligned}$$

   (b) Equilibrium Constraints

   $$\begin{aligned} 0 &= f(\bar{x}, -K_1\bar{\hat{x}} - K_2\bar{\sigma}) \\ 0 &= h(\bar{x}) - r \\ 0 &= (A - BK_1 - HC)\bar{\hat{x}} - BK_2\bar{\sigma} + Hh(\bar{x}) \end{aligned}$$

(c) For any fixed r of interest, the equilibrium conditions of step 2 are assumed to provide unique corresponding equilibrium values for $\bar{x}$ and $\bar{u}$, via solution of $n+m$ constraints on $n+m$ variables. The design hypotheses of step 4a guarantee that K_2 and $A - HC$ are nonsingular matrices. Consequently, for the given r there is a unique equilibrium state $(\bar{x}, \bar{\sigma}, \bar{\hat{x}})$ such that

$$\begin{aligned} \bar{\sigma} &= -K_2^{-1}\left(\bar{u} - K_1(A - HC)^{-1}(B\bar{u} + Hr)\right) \\ \bar{\hat{x}} &= -(A - HC)^{-1}(B\bar{u} + Hr) \end{aligned}$$

At this equilibrium state, the output matches the desired setpoint, i.e. $y = r$. Perturbations of (f, h) parameters affect the equilibrium state but do not destroy the desired property of zero steady-state error at the output.

(d) Jacobian Matrix Evaluation

$$\begin{aligned}
\left.\frac{\partial F_1}{\partial x}\right|_{(\bar{x},\bar{\sigma},\bar{\hat{x}})} &= \left.\frac{\partial f}{\partial x}\right|_{(\bar{x},\bar{u})} (I) = A \\
\left.\frac{\partial F_1}{\partial \sigma}\right|_{(\bar{x},\bar{\sigma},\bar{\hat{x}})} &= \left.\frac{\partial f}{\partial u}\right|_{(\bar{x},\bar{u})} (-K_2) = -BK_2 \\
\left.\frac{\partial F_1}{\partial \hat{x}}\right|_{(\bar{x},\bar{\sigma},\bar{\hat{x}})} &= \left.\frac{\partial f}{\partial u}\right|_{(\bar{x},\bar{u})} (-K_1) = -BK_1 \\
\left.\frac{\partial F_2}{\partial x}\right|_{(\bar{x},\bar{\sigma},\bar{\hat{x}})} &= \left.\frac{\partial h}{\partial x}\right|_{(\bar{x})} (I) = C \\
\left.\frac{\partial F_2}{\partial \sigma}\right|_{(\bar{x},\bar{\sigma},\bar{\hat{x}})} &= 0 \\
\left.\frac{\partial F_2}{\partial \hat{x}}\right|_{(\bar{x},\bar{\sigma},\bar{\hat{x}})} &= 0 \\
\left.\frac{\partial F_3}{\partial x}\right|_{(\bar{x},\bar{\sigma},\bar{\hat{x}})} &= H\left.\frac{\partial h}{\partial x}\right|_{(\bar{x})} (I) = HC \\
\left.\frac{\partial F_3}{\partial \sigma}\right|_{(\bar{x},\bar{\sigma},\bar{\hat{x}})} &= -BK_2 \\
\left.\frac{\partial F_3}{\partial \hat{x}}\right|_{(\bar{x},\bar{\sigma},\bar{\hat{x}})} &= A - BK_1 - HC
\end{aligned}$$

(e) The Jacobian matrix

$$A_{\text{cl}} = \begin{bmatrix} A & -BK_2 & -BK_1 \\ C & 0 & 0 \\ HC & -BK_2 & A - BK_1 - HC \end{bmatrix}$$

may be brought into the form

$$\begin{bmatrix} \mathcal{A} - \mathcal{B}\mathcal{K} & 0 \\ 0 & A - HC \end{bmatrix}$$

by elementary row and column operations. Hence, A_{cl} is Hurwitz by design (see step 4b). According to Lyapunov's Indirect Method, $(x, \sigma, \hat{x}) = (\bar{x}, \bar{\sigma}, \bar{\hat{x}})$ is an exponentially stable equilibrium point of the closed-loop system. At this equilibrium point, $y = r$ as desired.